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Gilbert Stengle

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1. Introduction

The purpose of this article is to exhibit a technique for solving certain problems in the asymptotic theory of differential equasions. We consider here, modulo inessential transformations, the most general second order equation susceptible to this technique. We have elected to give a detailed account of such a problem because the methods involved seem appropriate to a wide class of problems (see Stengle [5] for some results about n-th order equations).

We consider the equation

$$(1.1) \qquad \rho^{2n} \frac{d^2y}{dt^2} = a(t,\rho)y$$

where t and ρ are real variables ranging over $|t| \leq t_0$, $0 < \rho < \rho_0$ and $a(t,\rho)$ is C^∞ on the closure of this domain. In the case that $a(t,0)$ does not vanish, (1-1) falls within the scope of a systematic theory (see Turrittin [2]. However if $a(t,0)$ has isolated zeros, individual representatives of (1.1) become highly idiosyncratic and there exists a considerable literature devoted to the investigation of special cases (see Erdelyi [1]). Such problems are called "turning point" or "transition point" problems and the zeros of $a(t,0)$ are called "turning" or "transition" points. We describe a class of such problems which can be treated by adjoining a root $\lambda(t,\rho)$ of

$$(1.2) \qquad \lambda^2 - a = 0$$

to the resources of non-turning point theory. This class forms a

natural generalization of the class of problems which do not have turning points. Our results shed light on the difficult problem of classifying turning points, for which, we believe, no satisfactory definition has yet been given. We will meet a significant part of the difficulties below in classifying the singular behavior near (0,0) of $\lambda(t,\rho)$ and certain classes of functions arising from λ by the operations of differential algebra.

2. The Newton Polygon

In this section we state restrictive hypotheses which $a(t,\rho)$ must satisfy. We suppose H0. $a(t,0)$ has a zero of order m_o at $t = 0$ and $a(0,\rho)$ has a zero of order γ at $\rho = 0$.

Given a C^∞ function $\phi(t,\rho)$ let $\hat{\phi}$ denote the formal power series of ϕ at $t = \rho = 0$. Since every formal power series is the power series of some C^∞ function we also use circumflexed symbols to denote abstract formal power series or the formal product of such a series and a C^∞ function. Since $a_{m_o} \neq 0$, by the Weierstrass preparation theorem for formal power series

$$(2.1)\qquad \hat{a} = (t^{m_o} + \sum_{m=0}^{m_o-1} t^m \rho^{km} \hat{P}_m(\rho))\hat{U}_o(t,\rho) = \hat{P}_o\hat{U}_o$$

where $\hat{p}_m(\rho)$ is either a unit or identically 0 and $\hat{U}_o$ is a unit in the ring of formal power series.

We obtain the Newton polygon of (1-1) by plotting the points (k_m,m) for which $p_m \neq 0$ and forming the convex hull of these points and the set

$$S_o = \left\{(0,m) \mid m \geq m_o\right\} .$$

The boundary of this set is the Newton polygon $\underset{\sim}{N}$. The point $(\gamma,0)$ is an extreme point of this set and hence is a vertex of the boundary. We number the sides between S_o and $(\gamma,0)$ $S_1 \ldots S_p$. Let (k_j,μ_j) be the coordinates of the lower vertex of S_j. Let S_j be described by the equation

$$k + \delta_j m = \gamma_j$$

where δ_j, γ_j are positive rationals with least common denominator η_j.

It can be seen that the change of variables

$$(2.2)_j \qquad \begin{aligned} t &= \rho^{\delta_j} s \\ y(t,\rho) &= w(s,\rho) \end{aligned}$$

transforms (1-1) into

$$(2.3)_j \qquad \rho^{2n-2\delta_j-\gamma_j} \frac{d^2 w}{ds^2} = [s^{\mu_j} a_j(s) + \rho^{\frac{1}{\eta_j}} \beta_j(s, \rho^{\frac{1}{\eta_j}})]w$$

where $a_j(s)$ is the polynomial $U(0,0) \sum_{(k,m)\in S_j} \hat{p}_k(0)\ s^{m-\mu_j}$ for $j=1,2,\ldots,p$, $a_o(s) = s, ^{-\mu_o} a(s,0)$, and $\beta_j(s,\sigma)$ is C^∞.

We assume:

H1. $\qquad a_o(s) > 0 \qquad\qquad 0 \leq s \leq t_o$

$\qquad\quad a_j(s) > 0 \qquad\qquad 0 \leq s < \infty \qquad\qquad 1 \leq j \leq p.$

H2. $\qquad n - \delta_p - \frac{\gamma}{2}p \equiv \Delta > 0.$

We remark that if $p > 0$, $(2.3)_j$ is a turning point problem for $0 \leq j < p$ but $(2.3)_p$ is not. Hypothesis H2. implies however, that $(2.3)_p$ has a singular dependence on ρ. The case that (1.1) is not a turning point problem corresponds to the special case in which the Newton polygon consists of a single vertical ray.

3. The Connection Problem

Hypotheses H1. and H2. bring $(2.3)_j$ $0 < j \leq p$ within the scope of the standard theory if s is restricted to a domain of the form $0 < s_o \leq s_1$ for $0 < j < p$ or to $0 = s_o \leq s \leq s_1$ for $j = P$. The main result is that there exist solutions $w_j^{(1)}(t,\rho)$ $w_j^{(2)}(t,\rho)$ having asymptotic representations $\hat{w}_j^{(1)}$, $\hat{w}_j^{(2)}$ which are fundamental in the sense that solutions of $(2.3)_j$ on the same domain have asymptotic representations of the form $c_1(\rho)w_j^{(1)} + c_2(\rho)\hat{w}_j^{(2)}$. Moreover the $\hat{w}_j^{(i)}$ have the form

$$(3.1)_j \qquad \hat{w}_j^{(1)} = Q_j^{(i)} \exp q_j^{(i)} \qquad i=1,2$$

where $Q_j^{(i)}$ is an asymptotic power series in $\rho^{\frac{1}{\sigma_j}}$ with coefficients which are C^∞ functions of s, and $q_j^{(i)}$ is a polynomial in $\rho^{-\frac{1}{\sigma_j}}$ with similar coefficients.

Such results do not reveal to what extent the formal expressions obtained from $(3.1)_j$ by reversing the transformation $(2.2)_j$, namely $\hat{w}_j^{(i)}(t\rho^{-\delta_j},\rho)$, will describe the limiting behavior of the solutions $w_j^{(i)}(t\rho^{-\delta_j},\rho)$ in the case that t does not have the special form $t = s\rho^{\delta_j}$. Our hypotheses insure that for each ρ, the solution $w_j^{(i)}(t\rho^{-\delta_j},\rho)$ is the restriction to the domain $s_o\rho^{\delta_j} \leq t \leq s_1\rho^{\delta_j}$ of a "global" solution $y_j^{(i)}(t,\rho)$ on the domain

$0 \le t \le t_o$. However in general the pairs $y_j^{(1)}(t,\rho)$, $y_j^{(2)}(t,\rho)$ $j=0,1,\dots,p$, will be different. Among them must persist linear relations (depending on ρ). It is the case that the expressions $(3.1)_j$ provide only fragmentary knowledge of the asymptotic behavior of solutions unless we have an asymptotic description of these linear relations. We therefore ask:

1) To what extent do formal expressions $(3.1)_j$ provide asymptotic information if s depends on ρ?

2) What are the asymptotic linear relations among the pairs $y_j^{(1)}$, $y_j^{(2)}$?

We call this interrelated pair of questions the connection problem for the asymptotic solutions (3.1).

4. Formal Considerations

We begin with some definitions of a general nature:

Definition: Given a (t,ρ) set Ω, let $M(\Omega)$ be the ring of bounded functions on Ω.

Definition: Let Ω be a (t,ρ) set on which ρ^{-1} is unbounded. We say that a sequence f_k of functions on Ω is formally convergent to 0 if given any positive integer N there is a $k_o(N)$ such that

$$f^{(k)} \in \rho^{N} M(\Omega)$$

for all $k \ge k_o(N)$, and $\rho^{N_o} f_k \in M(\Omega)$ for some N_o and all k.

This notion of convergence leads directly to the ring $M^*(\Omega)$ of formally convergent series of functions on Ω. We use the symbol "$\diamond$" to denote equality in $M^*(\Omega)$. The archetype of a formally convergent series is a series of the form

$$\sum_{-N}^{\infty} f_k(t)\rho^k$$

where all but a finite number of the $f_k(t)$ are bounded on Ω.

Definition: We call a series of the above form a formal power series. If it is important to distinguish the special case $N = 0$, we use the term "proper formal power series".

Definition: Let f, f_k, $k=1,2,\ldots$, belong to $\rho^{-N_0} M\Omega$. We say that $\sum_{k=1}^{\infty} f_k$ is an asymptotic expansion of f on Ω if $\lim_{N \to \infty} (f - \sum_{k=1}^{N} f_k) \diamond 0$. We indicate this relationship by writing

$$f \sim \sum_{k=1}^{\infty} f_k.$$

Remarks. Evidently an asymptotic expansion is a formally convergent series. This notion of asymptotic expansion arises from the sequence of ideals $\rho^k M$ in M. It is possible to define more general kinds of expansions by introducing more general nested sequences of ideals but the preceding notion seems to include very many cases of interest in the theory of differential equations. Indeed our most delicate results involve other sequences of ideals, but since these depend so strongly on the individual characteristics of our problem it is most natural to let these ideals appear explicitly in the statement of our results.

We note that the preceding definitions do not exclude the possibility that t is a vector variable.

We now make definitions which are dictated by the particular exigencies of our problem.

Definition: Let $\Omega(t', s', \rho')$ be the (t,ρ) set $-s'\rho^{\delta_p} \leq t \leq t'$, $0 < \rho < \rho'$. Let t_o, ρ_o be positive, and $0 < s_o < 1$. Let $\Omega_o \equiv \Omega(t_o, s_o, \rho_o)$.

Definition: Let $z(t,\rho) = t + \rho^{\delta_p}$. Let $D = \frac{d}{dt}$. Let $\mathcal{M}(\Omega_o)$ be the set of functions f such that $z^k D^k f \in M(\Omega_o)$ $k=0,1,2,\ldots$. Let $\mathcal{M}^*(\Omega_o)$ be the set of formally convergent series of elements of $\mathcal{M}(\Omega_o)$.

Remarks: $M(\Omega_o)$ $\left(M^*(\Omega_o)\right)$ is a differential subring of $\mathcal{M}(\Omega_o)$ $\left(\mathcal{M}^*(\Omega_o)\right)$ with deviation zD (zD understood as termwise differentiation).

Definition: Let $P(t,\rho)$ be the polynomial (see (2.1))

$$t^{m_o} + \sum_{(m,k)\in S_1 \cup S_2 \ldots \cup S_p} \hat{P}_m(0)\, t^m \rho^k \ .$$

Remark: $P(t,\rho)$ is uniquely determined by $a(t,\rho)$.

Lemma 4a. For ρ_o, s_o sufficiently small, $a(t,\rho)$ can be represented in the following two ways:

1. $a = P_1 U_1 + A_1$

where P_1 is a monic polynomial in t of degree m_o, P_1 and U_1 are C^∞ on $|t| \leq t_o$, $0 \leq \rho \leq \rho_o$, $U_1 \neq 0$ on this domain, and

$$\hat{a} = \hat{P}_1 \hat{U}_1 \ .$$

2. $a = PU$

where P is defined above and U is a unit in $\mathcal{M}(\Omega_o)$.

Proof: We choose P_1 in the following way. We consider the equation $\hat{P}_o = 0$ as an algebraic equation for t with coefficients in the ring of formal ρ-power series. This has solutions

$\hat{t}_k^{(1)}(\rho^{\frac{1}{N}})$ $k = 1,2$ m_o in the ring of formal power series in some root $\rho^{\frac{1}{N}}$ of ρ. Let $t_k^{(1)}(s)$ be a C^{∞} function having $\hat{t}_k(s)$ as its formal power series.

Let

$$P_1(t,\rho) = \prod_{k=1}^{m_o}\left(t - t_k^{(1)}(\rho^{\frac{1}{N}})\right).$$

P_1 is easily seen to be a C^{∞} function of ρ with formal power series $\hat{P}_o$. Let U_1 be any C^{∞} function having $\hat{U}_o$ as its formal power series. Then $a_1 \equiv U - P_1U_1$ has formal power series 0.

2. We now examine the linear factors of P_1 and P. Under the transformation $t = s\rho^{\delta_j}$ P_1 and P assume the form

$$\rho^{\gamma_j} U_1^{-1}(0,0)\left\{s^{\mu_j}a_j(s) + \ldots\right\}$$

Where the dots indicate higher order terms in $\rho^{\frac{1}{\sigma_j}}$. These higher terms are in general different for $\hat{P}_1$ and P, but the leading parts agree since $\hat{P}_1$ and P agree in all terms which correspond to index pairs on sides $S_o - S_p$ of the Newton polygon. It follows (Semple and Kneebone[3]) that the roots of $\hat{P}$ and P as formal series are of the form

$$\rho^{\delta_{j_k}}(\zeta_k + \ldots) \quad k=1,\ldots,m_o$$

where ζ_k is a root of a_{j_k} and the dots indicate higher order terms in some fractional power of ρ. Since the series for the roots of $\hat{P}$ are convergent for small ρ, and since the series for the roots of P_1 and formal power series of the roots of P_1 by construction

we conclude that the roots $t_k^{(1)}(\rho)$ of P_1 and the roots $t_k(\rho)$ of P have the form

$$\left.\begin{matrix} t_k \\ \\ t_k^{(1)} \end{matrix}\right\} = \rho^{\delta j_k}[\zeta_k + o(1)] \qquad k=1,2,\ldots,m_o$$

By H2. each of the ζ_k are complex or negative real. For ρ_o, s_o, sufficiently small we can suppose that the distance in the complex plane of each $\zeta_k + o(1)$ from the subset $[-s_o,\infty]$ of the real axis is greater than some positive constant. This readily implies that the expressions

$$\left\{\frac{t-t_k}{t-t_k^{(1)}}\right\}^{\pm 1}, \quad z(t-t_k)^{-1}, \quad z(t-t_k^{(1)})^{-1}$$

are bounded on Ω_o and therefore generate a subring $\mathcal{N}(\Omega_o)$ of $\mathcal{M}(\Omega_o)$. Since (zD) applied to any of these generators is again an element of $N(\Omega_o)$, this subring is also a subring of $\mathcal{M}(\Omega_o)$. In particular $\frac{z^{m_o}}{P}$, $\frac{P_1}{P}$ and $\frac{P}{P_1}$ are elements of $\mathcal{M}(\Omega_o)$.

We write the representation of 1. in the form

$$a = P\left(\frac{P_1}{P}U_1 + \frac{A_1}{P}\right).$$

By 1. A_1 is a C^∞ function such that $\hat{A}_1 \equiv 0$. This implies $\left(D^{N_1}A_1\right)(|t| + \rho)^{-N_2}$ is bounded for all N_1, N_2 for all N_1, N_2 for $0 < \rho < \rho_o$, $|t| \leq t_o$, which implies that $A_1 \in z^N M$ for all N. Thus

$\frac{P_1}{P}U_1$ is a unit in M and $\frac{A_1}{P_0} = z\left(\frac{z^{m_0}}{P_0}\; z^{-m_0^{-1}}A_1\right) = za_1$ where $a_1 \in \mathcal{M}$.

For t' and ρ_0 sufficiently small, $U \equiv \frac{P_1U_1}{P} + 2A_1$ is also a unit. Thus on

$$\Omega(t', s_0, \rho_0)$$

$$a = UP$$

But for ρ_0 sufficiently small, or $\Omega(t_0, s_0, \rho_0) - \Omega(t', s_0, \rho_0)$, i.e. for $0 < t' \leq t \leq t_0$, our hypotheses imply that a and P are units in C^∞. Hence s_0 is U, and the conclusion of 2. follows.

Remark: The preceding result is a peculiar analogue of the Weierstrass preparation theorem for holomorphic functions. It cannot, of course be used to draw conclusions about the zeros of a since the result depends upon the fact that a has no zeros on Ω_0. Its significance lies in the fact that P characterizes the way in which a^{-1} is unbounded on Ω_0.

Lemma 4b. On Ω_0, $P > K_p^{\gamma_p}$ where K is a constant.

Proof: The factorization

$$P = \prod_{k=1}^{m_0}\left[t - \rho^{\delta_{j_k}}\zeta_k + o(1)\right]$$

and the fact that $a(0,\rho)$ has a zero of order $\gamma = \gamma_p$ at $\rho = 0$ implies

$$\sum_{k=1}^{m_0} \delta_{j_k} = \gamma_p = \gamma.$$

Thus $P = \rho^{\gamma_p}\prod_{k=1}^{m_0}\left[t\rho^{-\delta_{j_k}} - t_k^{(1)}(\rho)\rho^{-\delta_{j_k}}\right]$.

For $(t,\rho) \in \Omega_o$, $t\rho^{-\delta j_k}$ ranges over a subset of $[-s_o,\infty]$. In the proof of Lemma 4a, Ω_o was determined so that

$t_k^{(1)}(\rho)\rho^{-\delta j_k}$ had distance from this set greater than some constant K_1. Hence

$$P = |P| \geq \rho^{\gamma p} K_1^{m_o} = K\rho^{\gamma}$$

Lemma 4c. <u>Let</u> $I \equiv \rho^n P^{-1/2} z^{-1}$. <u>Then</u> $P^{1/2}$, I, $zP^{-1}\dot{P}$, $zI^{-1}\dot{I}$ <u>are all elements of</u> $\mathcal{M}(\Omega_o)$ <u>and</u> $\lim_{k\to} I^K \diamondsuit 0$.

Proof: Evidently $P^{1/2} \in M(\Omega_o)$. Also $zP^{-1}\dot{P}$ is an element of the subring $\mathcal{N}(\Omega_o) \subset \mathcal{M}(\Omega_o)$ used in the proof of the previous lemma. The identity

$$(zD)^{N+1}P^{1/2} = (zD)^N(\tfrac{1}{2}P^{1/2}[zP^{-1}\dot{P}])$$

and an induction argument shows that

$$(zD)^k P^{1/2} \in M \;\forall\; k \geq 0$$

i.e.

$$P^{1/2} \in \mathcal{M} .$$

By Lemma 4b

$$|P| > K_\lambda^{\gamma p}$$

where K is a positive constant. We can therefore estimate I:

$$|I| \leq K^{-1/2}\rho^{-\gamma \frac{\rho}{2}} + n^{-\delta}\rho(1-s_o)\ -1 \quad .$$

Since by Hypothesis H2

$$n^{-\gamma}\frac{\rho}{2} - \delta\rho = \triangle > 0, \quad I \in \rho^{\triangle} M \, .$$

This implies $\lim_{k \to \infty} I^k \Diamond 0$. Finally

$$zI^{-1}\dot{I} = -\frac{z}{2}\dot{P}P^{-1} - 1 \in \mathcal{M}$$

and as above induction shows $I \in \mathcal{M}$.

<u>Lemma 4d</u>. <u>The ideals in</u> $\mathcal{M}$, $P^{1/2}I^kM$ $k = 0,1,\ldots,$ <u>are closed under</u> zD.

Proof:
$$zD(P^{1/2}I^k\mathcal{M}) \in (zDP^{1/2})I^k\mathcal{M}$$
$$+ P^{1/2}(zDI^k)\mathcal{M} + P^{1/2}I^k(zD)$$
$$\in P^{1/2}(z\dot{P}P^{-1})I^k\mathcal{M} + P^{1/2}(I^kz\dot{I}I^{-1})\mathcal{M}$$
$$+ P^{1/2}I^k\mathcal{M}$$
$$\in P^{1/2}I^k\mathcal{M}.$$

5. Formal Solutions

We consider the Ricatti equation

(5.1) $$\rho^n\dot{r} + r^2 - a$$

related to (1.1) by the transformation

$$\rho^n\frac{\dot{y}}{y} = r.$$

The equation

$$\eta\dot{r} + r^2 - a = 0$$

has formal η-power series solutions

$$\sum_{k=0}^{\infty} \eta^k R_k$$

where R_o is a root, call it λ, of

$$R_o^2 - a = 0 ,$$

and

$$(5.2) \qquad R_{k+1} = -\frac{1}{2\lambda}(\dot{R}_k + \sum_{\substack{i+j=k+1 \\ i,j\geq 0}} R_i R_j).$$

It follows from (5.2) and from the relation $\dot{\lambda} = \dot{a}(2\lambda)^{-1}$ that R_k can be written in the form $R_k(\lambda,t,\rho)$ where R_k is a rational function of λ with coefficients in the ring generated by $a(t,\rho)$ and its t-derivatives.

We introduce the sequence ϕ_k according to the scheme

$$(5.3) \qquad \phi_o = 0$$

$$\rho^n \dot{\phi}_k + 2\lambda\phi_{k+1} = -\rho^n\dot{\lambda} - \phi_k^2.$$

This sequence can be described as "solution by formal successive approximation." We could use it to construct asymptotic solutions, but we give precedence to the sums $\sum_{o}^{k} R_j\rho^j$ which are, roughly speaking, the simplest expressions which approximate r to within the order of $\rho^{k+1} R_{k+1}$. However, we will use sequence (5.3) as a convenient intermediate basis of comparison in section 7.

<u>Theorem 1.</u>

i) $$\rho^{nk}R_k \in P^{1/2}\, I^k \mathcal{M}(\Omega_o)$$

ii) $$\phi_k - \sum_{j=1}^{k} \rho^{nj}R_j \in P^{1/2}\, I^{k+1}\mathcal{M}(\Omega_o)$$

iii) <u>The series</u> $\hat{r} \hat{\vee} \sum_{j=0}^{\infty} \rho^{nj} R_j$

<u>is a formal solution of (5.1)</u>.

iv) $\lim_{k \to \infty} (\lambda + \phi_k) \hat{\vee} \hat{r}$.

Proof:

i) By Lemma 4a

$$R_o = \pm P^{1/2} U^{1/2} \in P^{1/2} \mathcal{M}$$

since the square root of the unit U is in M_1. Hence i) is true for $k = 0$. Suppose it is true for $k \leq k_o$. By (5.2)

$$\rho^{(k_o+1)n} R_{k_o+1} = - \frac{\rho^n}{2\lambda z} (zD) \rho^{nk_o} R_{k_o}$$

$$- \frac{1}{2\lambda} \sum_{\substack{i+j=k_o+1 \\ i,j > 0}} (\rho^{in} R_i \rho^{jn} R_j)$$

$$\in I(zD) P^{1/2} I^{k_o} \mathcal{M}$$

$$+ P^{-1/2} \sum_{\substack{i+j=k_o+1 \\ i,j > 0}} P^{1/2} I^i P^{1/2} I^j \mathcal{M}$$

$$\in P^{1/2} I^{k_o+1} \mathcal{M}$$

ii) The statement is true for $k = 1$ since $\phi_1 = \rho^n R_1$. Suppose it is true for $k \leq k_o$. Then by (5.3)

$$\phi_{k+1} = -\rho^n \frac{\dot{\lambda}}{2\lambda} - \frac{1}{2\lambda}\dot{\phi}_k - \frac{1}{2\lambda}\phi_k^2$$

$$\in \rho^n R_1 - \frac{\rho^n}{2\lambda}\sum_{j=1}^{k_o} \rho^{nj}\dot{R}_j + \frac{\rho^n}{2\lambda z}(zD)P^{1/2}I^{k_o+1}\mathcal{M}$$

$$- \frac{1}{2\lambda}\Big(\sum_{i,j=1}^{k_o} \rho^{in}\rho^{jn}R_iR_j$$

$$+ [\sum_i^{k_o}\rho^{in}R_i]P^{1/2}I^{k_o+1}M + PI^{2k_o+2}\mathcal{M}\Big)$$

$$\in \rho^n R_1 + \sum_{k=1}^{k_o+1} - \rho^{n(k+1)}\,[\dot{R}_k + \sum_{\substack{i+j=k+1\\ i,j>0}} R_iR_j]$$

$$+ P^{-1/2}\sum_{\substack{i+j>k_o+1\\ i,j<2k_o}} P^{1/2}I^iP^{1/2}I^j$$

$$+ P^{1/2}I^{k_o+2}M + P^{1/2}I^{2k_o+2}\mathcal{M}$$

$$\in \rho^n R_1 + \sum_{k=1}^{k_o+1} \rho^{n(k+1)}R_{k+1} + P^{1/2}I^{k_o+2}\mathcal{M}$$

$$\sum_{k=1}^{k_o+2} \rho^{kn}R_k + P^{1/2}I^{k_o+2}\mathcal{M} .$$

iii) Since $\lim\limits_{k\to\infty} I^k \mathrel{\hat{\vee}} 0$, i) implies that $\sum_{j=1}^{k}\rho^{nj}R_j$ and $\rho^n D\sum_{j=1}^{k}\rho^j R_j$, $k=1,2,\ldots$, are formally convergent sequences substitution of which into (5.1) leads to a sequence formally convergent to 0.

iv) By ii) $\lim\limits_{k\to\infty} \phi_k - \sum_{j=1}^{k}\rho^{nj}R_j \mathrel{\hat{\vee}} 0.$

Since $\hat{r} - \lambda$ is the limit of the sum, iv) follows.

6. Construction of Solutions

We now solve (5.1) by successive approximations. Let

(6.1) $$r = \lambda + \psi.$$

Then ψ must satisfy

$$\psi = -\int_c^t \exp\left\{-2\rho^{-n}\int_s^t \lambda d\sigma\right\}(+\lambda' + \rho^{-n}\psi^2)ds.$$

We choose c to be $-s_o\rho^{\delta p}$ if $\lambda = +a^{1/2}$ and t_o if $\lambda = -a^{1/2}$. We distinguish those two cases by the subscripts "$\pm$" and indicate the integral equation by

(6.2) $$\psi = \psi_{\pm}^{(1)} + \rho^{-n}L_{\pm}(\psi^2)$$

We define the sequences

$$\psi_{\pm}^{(o)} = 0$$

(6.3)
$$\psi_{\pm}^{(k+1)} = \psi_{\pm}^{(1)} + \rho^{-n}L_{\pm}\left([\psi_{\pm}^{(k)}]^2\right) \quad k=0,1,2,\ldots \quad .$$

<u>Definition:</u> <u>For</u> $u \in M(\Omega_o)$ <u>let</u>

$$||u|| \equiv \sup_{-s_o\rho^{\delta p}\leq t\leq t_o}|u|$$

<u>Lemma 6a</u>: For $u \in \mathfrak{M}(\Omega_o)$, the image of the ideal $u\mathfrak{M}(\Omega_o)$ under $L_{\pm}$ satisfies

$$L_{\pm}\left(u\mathcal{M}(\Omega_o)\right) \subset ||zuI||\,M(\Omega_o) + L_{\pm}\left([u+z\dot{u}]\,I\mathcal{M}(\Omega_o)\right).$$

Proof: Integration by parts.

<u>Lemma 6b</u>: <u>If</u> $u \in \mathcal{M}(\Omega_o)$ <u>and</u> $z\,\frac{\dot{u}}{u} \in \mathcal{M}(\Omega_o)$, <u>then</u>

$$L_{\pm}\left(u\mathcal{M}(\Omega_o)\right) \subset ||zuI||\,M(\Omega_o).$$

Proof: If $z\,\frac{\dot{u}}{u} \in \mathcal{M}$ then $[u+z\dot{u}]\,I\,\mathcal{M} \subset uI\mathcal{M}$.

Also $z\frac{(u\dot{I})}{uI} \in \mathcal{M}$. Hence by repeated application of Lemma 6a

$$L_{\pm}(uM) \subset ||zuI||\,M + L_{\pm}(uI\mathcal{M})$$

$$\subset ||zuI||\,M + ||zuI^2||\,M + L_{\pm}(uI^2\mathcal{M})$$

$$\subset \left\{||zuI|| + ||zuI^2|| + ||zuI^k||\right\} M + L_{\pm}(uI^k\mathcal{M})$$

$$\subset ||zuI||\,M + L_{\pm}(uI^k\mathcal{M}) .$$

But for k sufficiently large $I^k\mathcal{M} \subset zIM$.

Hence $L_{\pm}(u\mathcal{M}) \subset ||zuI||\,M + L_{\pm}(zuIM)$

$$\subset ||zuI||\,M.$$

<u>Lemma 6c</u>: $||L_{\pm}(u)|| < C\,\rho^{n-\gamma/2}||u||$, <u>where</u> C <u>is a constant</u>.

Proof: Since $||L_{\pm}(u)|| \le ||u||\ ||L_{\pm}(1)||$ it suffices to show $L_{\pm}(\rho^{\gamma/2-n}) \in M$. By Lemma 6b

$$L_{\pm}(\rho^{\gamma/2-n}) \in \|\ ^{\gamma/2-n}\ zI\|M$$

$$\in \|\frac{\rho^{\gamma/2}}{p^{1/2}}\|M$$

Lemma 4b then implies:

$$L_{\pm}(\rho^{\gamma/2-n}) \in M.$$

Lemma 6d: $\psi_{\pm}^{(1)} \in \rho^{\Delta + \gamma/2} M(\Omega_o)$.

Proof: $\psi_{\pm}^{(1)} = L_{\pm}(\dot\lambda)$. Integrating by parts

$$L_{\pm}(\dot\lambda) \in \|z\dot\lambda I\|M + L([\dot\lambda + z\dot\lambda]I\mathcal{M}).$$

But $z\dot\lambda I \in \frac{\rho^n}{z}\mathcal{M}$ and $(\dot\lambda + z\dot\lambda)I \in \frac{\rho^n}{z^2}\mathcal{M}$.

Hence

$$\psi_{\pm}^{(1)} \in \|\frac{\rho^n}{z}\|M + L_{\pm}\left(\frac{\rho^n}{z^2}\mathcal{M}\right)$$

By Lemma 6b

$$\psi_{\pm}^{(1)} \in \|\frac{\rho^n}{z}\|M + \|\frac{\rho^n}{z}\ I\|M$$

$$\in \rho^{n-\delta}P_M = \rho^{\Delta + \gamma/2}M.$$

Theorem 2. For ρ_o sufficiently small the limits

$\psi_{\pm} = \lim_{k \to \infty} \psi_{\pm}^{(k)}$ exist and are solutions of (6.2). Moreover

(6.4) $$\|\psi_{\pm}^{(k+1)} - \psi_{\pm}^{(k)}\| < \left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2 + k\delta}.$$

Proof: By Lemmas 6c, 6d and (6.3) the sequences $\|\psi_{\pm}^{(k)}\|$ are dominated, for some constant C by $\Phi_o = 0$

$$\Phi_{k+1} = \frac{C}{2}\left(\rho^{\Delta + \gamma/2} + \rho^{-\gamma/2}\Phi_k^2\right) .$$

The last equation can be written

$$\frac{\Phi_{k+1}}{C(2\rho_o)^{\Delta+\gamma/2}\left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2}}$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{\rho}{2\rho_o}\right)^{\Delta}(2\rho_o)^{\Delta}C^2\left[\frac{\Phi_k}{C(2\rho_o)^{\Delta+\gamma/2}\left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2}}\right]^2 .$$

Choosing ρ_o so small that $C(2\rho_o)^{\Delta+\gamma/2} < 1$, $C^2(2\rho_o)^{\Delta} < 1$, we have,

$$\frac{\Phi_{k+1}}{\left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2}} < \frac{1}{2} + \frac{1}{2}\left(\frac{\rho}{2\rho_o}\right)^{\Delta}\left[\frac{\Phi_k}{\left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2}}\right]^2 .$$

It follows by induction that

$$\Phi_k < \left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2} .$$

which implies

$$(6.5) \qquad ||\Phi_{\pm}^k|| < \left(\frac{\rho}{2\rho_o}\right)^{\Delta+\gamma/2} .$$

Again by (6.3) and Lemma 6c, for some C

$$||\psi_{\pm}^{(k+1)} - \psi_{\pm}^{(k)}|| \leq C\,\rho^{-\gamma/2}\left\{||\psi_{\pm}^{(k)}|| + ||\psi_{\pm}^{(k-1)}||\right\}\left\{||\psi_{\pm}^{(k)} - \psi_{\pm}^{(k-1)}||\right\}$$

$$\leq 2C\,(2\rho_o)^{\Delta}\left(\frac{\rho}{2\rho_o}\right)^{\Delta}||\psi_{\pm}^{(k-1)}|| .$$

Choosing ρ_o so that $2C(2\rho_o)^{\Delta} < 1$ we have

$$(6.6) \qquad ||\psi_{\pm}^{(k+1)} - \psi^{(k)}|| \leq \left(\frac{\rho}{2\rho_0}\right)^{\Delta} ||\psi_{\pm}^{(k)} - \psi_{\pm}^{(k-1)}|| \quad .$$

This insures the uniform convergence of $\psi_{\pm}^{(k)}$ to solutions $\psi_{\pm}$ of (6.2). Inequalities (6.6) and (5.6) imply (6.4).

7. Formal Solutions are Asymptotic Solutions

We first establish some estimates involving $L_{\pm}$ which do not involve the uniform norm $|| \; ||$.

Definition: Let $\mathcal{E}_{\pm}$ denote the ring generated by $M(\Omega_0)$ and

$$\rho^{-k} \exp -2\rho^{-n} \int_{C_{\pm}}^{t} \lambda(s)ds \qquad k = 1,2,\ldots, \quad .$$

Lemma 7a: If $u \in z^{-1}P^{1/2}I^{j}\mathcal{M}(\Omega_0)$, then for each

$$L_{\pm}(u) \in P^{1/2}I^{j+1}\mathcal{M}(\Omega_0) + \mathcal{E}_{\pm} + \rho^{g}M(\Omega_0).$$

Proof: Repeated integration by parts shows

$$L_{\pm}(u) = (u^{(1)} \ldots + u^{(N)}) \exp -2\rho^{-n} \int_{s}^{t} \lambda(\sigma)d\sigma \Big]_{C_{\pm}}^{t} + L_{\pm}(v^{(N)})$$

where
$$u^{(k)} \in P^{1/2}I^{j+k}\mathcal{M} \quad k = 1,\ldots,N$$
$$v^{(N)} \in z^{-1}P^{1/2}I^{j+N}\mathcal{M} \quad .$$

This implies

$$L_{\pm}(u) \in P^{1/2}I^{j+1}\mathcal{M} + \mathcal{E}_{\pm} + L_{\pm}(z^{-1}P^{1/2}I^{j+N}\mathcal{M})$$

But for N sufficiently large $z^{-1}P^{1/2}I^{j+N}\mathcal{M} \subset \rho^{g}\mathcal{M}$. Since

$L_{\pm}(\rho^{\varepsilon}\mathcal{M}) \subset \rho^{\varepsilon}\mathcal{M}$ the conclusion follows.

Lemma 7b. $\rho^{-n}L_{\pm}(\mathcal{E}_{\pm}) \subset \mathcal{E}_{\pm}$.

Proof: Direct verification.

Theorem 3: Let the sequence $\phi^{(k)}$ be defined by (5.3). Let the sequence $\psi^{(k)}$ be defined by (6.3). Then for each $\varepsilon > 0$

$$\psi_{\pm}^{(k)} - \phi^{(k)} \in P^{1/2}I^{k+1}\mathcal{M}(\Omega_o) + \mathcal{E}_{\pm} + \rho^{\varepsilon}M(\Omega_o).$$

Proof: Since $\psi^{(o)} = \phi^{(o)} = 0$ the statement is true for $k = 0$. Suppose it is true for $k = N$. The recursion formula of (5.3) can be written

$$\rho^{n}\dot{\phi}_{N+1} + 2\lambda\ \phi_{N+1} = \rho^{n}(\dot{\phi}_{N+1} - \dot{\phi}_{N}) - \phi_{N}^{2} + \rho^{n}\dot{\lambda} \ .$$

This implies

$$\phi_{N+L} = \psi_{\pm}^{(1)} + L_{\pm}(\dot{\phi}_{N+1} - \dot{\phi} - \rho^{n}\phi_{N}^{2}) + \phi_{N+1}(c_{\pm})\exp - 2\rho^{-n}\int_{c_{\pm}}^{t}\lambda(s)ds \ .$$

Subtracting the last equation from (6.3) we obtain

$$\begin{aligned}\psi_{N+1} - \phi_{N+1} &= L_{\pm}\left[\dot{\phi}_{N+1} - \dot{\phi}_{N} + \rho^{-n}(\psi_{N}-\phi_{N})(\psi_{N}+\phi_{N})\right] \\ &\quad + \phi_{N+1}(c_{\pm})\exp - 2\rho^{-n}\int_{c_{\pm}}^{t}\lambda(s)ds.\end{aligned}$$

By Theorem 1 $\phi_{N+1} - \phi_N \in P^{1/2}I^{N+1}\mathfrak{M}$ and $\phi_N \in P^{1/2}I\mathfrak{M}$. Using the estimate of the induction hypothesis, for each g

$$\psi_{N+1} - \phi_{N+1} \in L_{\pm}(z^{-1}P^{1/2}I^{N+1}\mathfrak{M})$$

$$+ L_{\pm}\left(\rho^{-n}\left(\left[P^{1/2}I^{N+1}\mathfrak{M} + \mathcal{E}_{\pm} + \rho^{g}M\right]\right.\right.$$

$$\left.\left.\left[P^{1/2}I\mathfrak{M} + \mathcal{E}_{\pm} + \rho^{g}M\right]\right)\right) + \mathcal{E}_{\pm}$$

$$\in L_{\pm}(z^{-1}P^{1/2}I^{N+1}\mathfrak{M}) + L_{\pm}(\rho^{g-n}M) + L_{\pm}(\mathcal{E}_{\pm}) + \mathcal{E}_{\pm} \quad .$$

By Lemmas 7a and 7b, this implies for any g'

$$\psi_{N+1} - \phi_{N+1} \in P^{1/2}I^{N+2}\mathfrak{M} + \rho^{g'}M + \rho^{g-n}M + \mathcal{E}_{\pm}.$$

Since g' and g-n are arbitrary the conclusion follows.

<u>Corollary</u>: $\psi_k - \phi_k \in P^{1/2}I^{k+1}M(\Omega_o) + \mathcal{E}_{\pm}$.

Proof: For g(k) sufficiently large $\rho^{g(k)}M \in P^{1/2}I^{k+1}M$.

Definition: <u>Let</u> $\Omega_1 = \Omega(t_1, 0, \rho_1)$ <u>where</u> $0 < t_1 < t_o$ <u>and</u> $\rho_1 \leq \rho_o$.

<u>Lemma 7c</u>. <u>The restriction of</u> $\mathcal{E}_{\pm}$ <u>to</u> Ω_1 <u>is contained in</u> $\rho^{g}\mathfrak{M}(\Omega_1)$ <u>for each</u> g.

Proof: By Lemma 4b

$$\left| \int_{c_{\pm}}^{t} P^{1/2}\, ds \right| > K^{1/2}\rho^{\gamma/2}\, |t - c_{\pm}|.$$

For $t \in \Omega_1$ this implies

$$\exp + \rho^{-n}\int_{c_\pm}^{t} p^{1/2}ds < \exp - K'\rho^{-n+\gamma/2} + \delta\rho$$
$$< \exp - K'\rho^{-\Delta} \quad .$$

By H2., $\Delta > 0$. Since $\rho^{-\varepsilon}\exp -K'\rho^{-\Delta} \in M(\Omega_1)$ for each g the conclusion follows.

Corollary: On Ω_1

$$\psi_\pm^{(k)} - \phi_k \in p^{1/2}I^{k+1}M(\Omega_1).$$

Theorem 4. Let $r_\pm$ be the solutions of (5.1) given by

$$r_\pm = \pm\lambda + \psi_\pm \ .$$

Then for $(t,\rho) \in \Omega_1$.

(7.1) $$r_\pm - \sum_{j=0}^{k} \rho^{jn}R_j(\pm\lambda, t, \rho) \in p^{1/2}I^{k+1}M(\Omega_1).$$

Proof: The above difference can be written

$$(\psi_\pm - \psi_\pm^{(N)}) + (\psi_\pm^{(N)} - \phi_N) + \phi_N - \sum_{j=1}^{N} \rho^{jn}R_j + \sum_{j=k+1}^{N} \rho^{jn}R_j.$$

By Theorem 2 $\psi_\pm - \psi_\pm^{(N)} \in \rho^{\Delta+\gamma/2+N\Delta}M(\Omega_1)$.

By Theorem 3 $\psi_\pm^{(N)} - \phi_N \in p^{1/2}I^{N+1}M(\Omega_1)$.

By Theorem 1 $\phi_N - \sum_{j=1}^{N} \rho^{j}R_j \in p^{1/2}I^{N+1}M(\Omega_1)$ and

$$\sum_{j=k+1}^{N} \rho^{jN}R_j \in p^{1/2}I^{k+1}M(\Omega_1).$$

Supposing without loss of generality that $N \geq K$ we have

$$r_{\pm} - \sum_{j=0}^{k} \rho^{jn} R_j \in \rho^{\Delta+\gamma/2+N\Delta} M_1 + P^{1/2} I^{k+1} M_1 .$$

Choosing N so large that $\rho^{\Delta+\gamma/2+N\Delta} \in P^{1/2} I^{k+1} M_1$ the conclusion follows.

Corollary: Relation (7.1) implies the weaker statement that on $M(\Omega_1)$ $r_{\pm} \sim \sum_{j=0}^{\infty} \rho^{nj} R_j(\pm\lambda, t, \rho)$.

Lemma 7d. If $u \in P^{1/2} I^{k+1} M$, $k > 1$ then $\rho^{-n} \int_t^{t_0} u ds \in I^k M_1$.

Proof: $$\rho^{-n} \int_t^{t_0} u ds \in \int_t^{t_0} \rho^{-nk} P^{-k/2} z^{-k-1} M ds$$

$$\in \rho^{nk} P(t)^{-k/2} \int_t^{t_0} \left(\frac{P(t)}{P(s)}\right)^{k/2} z^{-k-1} M ds$$

$$\in \rho^{nk} P^{-k/2} \int_t^{t_0} z^{-k-1} M ds .$$

Since z is positive this implies

$$\rho^{-n} \int_t^{t_0} u ds \in \rho^{nk} P^{-1/2} \left(\int_t^{t_0} z^{-k-1} ds \right) M \subset I^k M .$$

Corollary: For $k > 1$.

$$(7.2) \quad \rho^{-n} \int_t^{t_0} r_{\pm} ds - \sum_{j=1}^{k+1} \rho^{(k+1)} \int_t^{t_0} R_k(\pm\lambda, s, \rho) ds \in I^{k+1} M_1 .$$

We have also determined solutions of (1-1) which we take to be those described by the fundamental matrix

$$(7.3)\quad \underset{\sim}{W} \equiv \begin{bmatrix} y_+ & y_- \\ \dot{y}_+ & \dot{y}_- \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_+ & r_- \end{bmatrix} \exp - \rho^{-n} \int_t^{t_0} \begin{pmatrix} r_+ & 0 \\ 0 & r_- \end{pmatrix} ds.$$

Since it is asymptotic formulas for solutions which are of greatest interest, we refer to these solutions only indirectly in the following Theorem, merely asserting that such solutions exist.

Theorem 5. (Asymptotic Solution of Equation (1.1))

Let the equation (1.1) *satisfy conditions* H0 *through* H2. *Let the sequence* $R_k(\lambda, t, \rho)$ *be given by* (5.2). *Then for* $0 \leq t \leq t_0$ *and for* ρ_1 *sufficiently small, on the domain* $\Omega_1 = \{(t,\rho):\ 0 \leq t \leq t_1 < t_0,\ 0 < \rho < \rho_1\}$, *there exists a fundamental pair of solutions* $Y_\pm$ *for which the matrix*

$$\underset{\sim}{Y} \equiv \begin{bmatrix} y_+ & y_- \\ \dot{y}_+ & \dot{y}_- \end{bmatrix}$$

satisfies

$$(7.4)\qquad \underset{\sim}{Y} \exp \rho^{-n} \int_t^{t_1} \begin{bmatrix} R_0(+\lambda) + \rho^n R_1(+\lambda) & \\ & R_0(-\lambda) + \rho^n R_1(-\lambda) \end{bmatrix}$$

$$- \sum_{j=0}^{k} \rho^{jn} \begin{bmatrix} R_j(+\lambda) & \\ & R_j(-\lambda) \end{bmatrix} \exp \rho^{-n} \sum_{j=2}^{k+1} \int_t^{t_1} \begin{bmatrix} R_j(+\lambda) & \\ & R_j(-\lambda) \end{bmatrix} ds$$

$$\in I^{k+1} \begin{bmatrix} 1 & 0 \\ 0 & P^{1/2} \end{bmatrix} M_k.$$

<u>where M_k has elements in $M(\Omega_1)$ for each</u> k.

Proof: We show that for ρ_1 sufficiently small the matrix $\underset{\sim}{W}$ of (6.2) is non-singular. Theorem 2 implies $\psi_{\pm} \in \rho^{\Delta+\gamma/2}M$ which in turn implies $\lambda^{-1}\psi_{\pm} \in \rho^{\Delta}M$. Thus

$$\begin{pmatrix} 1 & 1 \\ r_+ & r_- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho^{\Delta}H \right]$$

where H has elements in $M(\Omega_0)$. It follows that for ρ_1 small $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho^{\delta}H$ is non-singular and hence $\underset{\sim}{W}$ is non-singular. Equation (7.4) is a direct consequence of (7.1) and (7.3).

Remark: Since t_1 is any number less than the original t_0 this result is global in t.

<u>8. Solution of the Connection Problem</u>

We observe that Theorem 5 gives an asymptotic description of $\overline{Y}_{\pm}(t.\rho)$ uniformly on the t-domain $0 \leq t \leq t_1$. The form of the remainder in (7.4) shows that our asymptotic series behave, roughly speaking, like asymptotic power series in $I(t,\rho)$. For example if t is restricted to the domain $0 \leq t_2 \leq t \leq t_1$, then $I(t,\rho) = O(\rho^n)$, while at the other extreme if $t = 0$, $I(t,\rho) = O(\rho^{\Delta})$. Our object is now to make more specific assumptions about t and by inserting asymptotic expansions for individual terms in (7.4) to obtain expansions of the form given in Section 3.

We will show that we have to a large extent reduced the problem of determining the asymptotic behavior of the solutions of (1.1) to the elementary problem of determining the asymptotic behavior of the roots of $\lambda^2 - a = 0$. We attack the latter problem in the following definition ahd Theorem.

Definition: Let $\gamma_1, \gamma_2, \ldots, \gamma_p$ be real numbers such that

$$0 = \delta_o < \gamma_1 < \delta_1 < \gamma_2 \quad < \gamma_p < \delta_p.$$

Let $\tau_{p+1} = \sigma_p = 0 < \tau_p < \sigma_{p-1} < \tau_{p-1} < \sigma_{p-1} \cdots < \sigma_o = \tau_o = \tau_1$

be a subdivision of $[0, t_1]$, where for $1 \leq k \leq p$

$$\tau_j = s_j \rho^{r_j} \quad , \quad \sigma_j = s_j' \rho^{\delta_j}$$

and s_j, s_j' are positive variables. For ρ_1 sufficiently small we can suppose that s_j, s_j' range over a closed interval J of positive numbers containing 1.

Let I_j be the domain $\tau_{j+1} \leq t \leq \tau_j$, $0 < \rho \leq \rho_1$, $s_j, s_{j+1} \in J$ for $0 \leq j \leq p$. Let I_j', I_j'' be the domains $\tau_{j+1} \leq t \leq \sigma_j$, $0 < \rho \leq \rho_1, s_j', s_{j+1} \in J$ and $\sigma_j \leq t \leq \tau_j$, $0 < \rho \leq \rho_1$, $s_j, s_j' \in J$ respectively. Let J^* be the domain $0 < \rho \leq \rho_1$, $s_j, s_j' \in J$, $j = 1, 2, \ldots, p$.

Theorem 6. For ρ_1 sufficiently small, on I_j

$$(8.1) \quad \mathcal{Q}_k \equiv a(t,\rho) \left\{ \rho^{\gamma_j} (t\rho^{-\delta_j})^{\mu_j} a_j(t\rho^{-\delta_j}) \right\}^{-1} - 1 \in \rho^{\Delta_j} M(I_j)$$

where

$$\Delta_j = \begin{cases} \min(1, \delta_1 - \gamma_1, \gamma_1) & j = 0 \\ \min(1, \delta_{j+1} - \gamma_{j+1}, \gamma_j - \delta_{j-1}, \gamma_j) & 0 < j < p \\ \min(1, \gamma_p - \delta_{p-1}, \gamma_p) & j = p. \end{cases}$$

Proof: Suppose $0 < j < \rho$. By Lemma 4a

$$a_1 = P_1 U_1 + A_1$$

where $A_1 \in z^N M(\Omega_1)$ for each N. The polynomial $s^{\mu_j} a_j(s)$ has a pole of order μ_{j-1} at ∞ and a zero of order μ_j at 0 and has no zeros for $s > 0$. It follows that

$$s^{-\mu_j} a_j^{-1}(s) < \begin{cases} K\ s^{-\mu_j} & 0 < s \leq s_j' \\ K\ s^{-\mu_{j-1}} & s_j' < s < \infty \end{cases}$$

where K is a constant. This implies

$$(8.2) \qquad \left\{ \rho^{\delta_j} (t\rho^{-\delta_j})^{\mu_j} a_j (t\rho^{-\delta_j}) \right\}^{-1}$$

is contained in the sets

$$\begin{cases} \rho^{-\gamma_j} (t\rho^{-\delta_j})^{\mu_j} M(I_j') = \rho^{-k_j} t^{\mu_j} M(I_j') \\ \rho^{-\gamma_j} (t\rho^{-\delta_j})^{-\mu_{j-1}} M I_j'' = \rho^{-k_{j-1}} t^{-\mu_{j-1}} M(I_j''). \end{cases}$$

On I_j, $z \in \rho^{\gamma_j} M(I_j)$ and (8.2) implies the weaker estimate

$$(t\rho^{-\delta_j})^{-\mu_j} a_j^{-1} (t\rho^{-\delta_j}) \in (t\rho^{-\delta_j})^{\mu_j} M(I_j)$$

$$\in \rho^{-(\delta_{j+1} - \delta_j)\mu_j} M(I_j).$$

Hence

$$\rho^{-\gamma_j}(t\rho^{-\delta_j})^{-\mu_j}a_j^{-1}(t\rho^{-\delta_j})_{A_1} \in \rho^{-\gamma_j-(\delta_{j+1}-\delta_j)\mu_j+N\delta_{j-1}}M(I_j).$$

Since $\gamma_j > 0$ and N is arbitrary, the above expression is ~ 0 on I_j and it suffices to consider

$$(8.3) \quad Q_j' = P_1U_1\left\{\rho^{\gamma_j}(t\rho^{-\delta_j})^{\mu_j}a_j(t\rho^{-\delta_j})\right\}^{-1} -1.$$

It is easily seen that

$$\rho^{\gamma_j}(t\rho^{-\delta_j})^{\mu_j}a_j(t\rho^{-\delta_j}) = U_1(0,0) \sum_{(k,m)\in S_j} p_m(0)t^m\rho^k.$$

The functions $p_m(\rho)$, $U_1(t,\rho)$ can be represented in the form

$$p_m(\rho) = p_m(0) + \rho\overline{p}_m \qquad , \overline{p}_m \in C^\infty$$

$$U_1(t,\rho) = U_1(0,0) + tU_2 + \rho U_3 \, , \, U_2,U_3 \in C^\infty .$$

Inserting these representations in (8.3)

$$Q_j' = \frac{\sum_{(k,m)\in S_j}\left\{(U_1(0,0)+tU_2+\rho U_3)\rho^k t^m(p_m(0)+\rho\overline{p_m})-U(0,0)p_m(0)\rho^k t^m\right\}}{U_1(0,0)\sum_{(k,m)\in S_j} p_m(0)\rho^k t^m}$$

$$+ \frac{U_1(t,\rho)\sum_{(k,m)\in N-S_j} p_m\rho^k t^m}{U_1(0,0)\sum_{(k,m)\in S_j} p_m(0)\rho^k t^m}$$

This representation implies

$$(8.3) \quad Q_j' \in \frac{\sum_{(k,m)\in N-S_j}\rho^k t^m M(I_j) + \sum_{(k,m)\in S_j}\left\{\rho^k t^{m+1}M(I_j)+\rho^{k+1}t^m M(I_j)\right\}}{U_1(0,0)\sum_{(k,m)\in S_j} p_m(0)\rho^k t^m} .$$

This representation implies

$$(8.3)\quad Q_j' \in \frac{\sum_{(k,m)\in \underset{\sim}{N}-S_j} \rho^{k} t^{m} M(I_j) + \sum_{(k,m)\in S_j} \{\rho^{k} t^{m+1} M(I_j) + \rho^{k+1} t^{m} M(I_j)\}}{U_1(0,0) \sum_{(k,m)\in S_j} P_m(0)\rho^{k} t^{m}} .$$

We consider the sum $\sum_{(k,m)\in N-S_j} \rho^{k} t^{m} M(I_j)$. Suppose (k,m) is on a side of $\underset{\sim}{N}$ to the left of S_j. Then $\rho^{k} t^{m} (\rho^{\kappa_{j-1}+\delta_{j-1}} t^{-\mu_{j-1}-1})$ has as a factor the non-negative power $t^{m-\mu_{j-1}-1}$ and is therefore an element of

$$\rho^{k-\kappa_{j-1}+\delta_{j-1}} (\rho^{\delta_{j-1}})^{m-\mu_{j-1}-1} M(I_j)$$

$$= \rho^{k+m\delta_{j-1} - \kappa_{j-1} - \delta_{j-1}\mu_{j-1}} M(I_j)$$

$$= \rho^{k+m\delta_{j-1} - \gamma_{j-1}} M(I_j).$$

But the point (k,m) is not below the line describing S_{j-1}. Hence $k + \delta_{j-1} m - \gamma_{j-1} \geq 0$ and $\rho^{k+m\delta_{j-1}-\gamma_{j-1}} M(I_j) \subset M(I_j)$.

Thus for $(k,m) \in S_0 \cup S, \ldots, \cup S_{j-1}$.

$$(8.4)\qquad t^{m}\rho^{k} \in \rho^{\kappa_{j-1}-\delta_{j-1}} t^{\mu_{j-1}+1} M(I_j).$$

Similarly for (k,m) on a side to the right of S_j

$$(8.5)\qquad t^{m}\rho^{k} \in \rho^{\kappa_j+\delta_{j+1}} t^{\mu_j-1} M(I_j).$$

Relations (8.3), (8.4), (8.5) imply

$$(8.6)\quad Q_j' \in \frac{\sum_{(k,m)\in S_j} \{\rho^{k+1} t^{m} M(I_j) + \rho^{k} t^{m+1} M(I_j)\}}{U_1(0,0) \sum_{(k,m)\in S} P_m(0)\rho^{k} t^{m}} + \frac{\rho^{\kappa_{j-1}-\delta_{j-1}} t^{\mu_{j-1}+1} M(I_j) + \rho^{\kappa_j+\delta_{j+1}} t^{\mu_j-1} M(I_j)}{U_1(0,0) \sum_{(k,m)\in S_j} P_m(0)\rho^{k} t^{m}}$$

On I_j' (8.3) and (8.6) imply

$$Q_j' \in \sum_{(k,m)\in S_j} \left\{ \rho^{k-\kappa_j+1} t^{m-\mu_j} M(I_j') + \rho^{k-\kappa_j} t^{m-\mu_j+1} M(I_j')' \right\}$$

$$+ \rho^{\kappa_{j-1}-\delta_{j-1}-\kappa_j} t^{\mu_{j-1}-\mu_j+1} M(I_j') + \rho^{\delta_{j+1}} t^{-1} M(I_j') \quad .$$

On I_j' , $t \in \rho^{\delta_j} M(I_j')$ and $t^{-1} \in \rho^{-r_{j+1}} M(I_j')$. Hence

$$Q_j' \in \sum_{(k,m)\in S_j} \rho^{k-\kappa_j+1 + (m-\mu_j)\delta_j} M(I_j')$$

$$+ \sum_{(k,m)\in S_j} \rho^{k-\kappa_j + (m-\mu_j+1)\delta_j} M(I_j')$$

$$+ \rho^{\kappa_{j-1}-\delta_{j-1}-\kappa_j + (\mu_{j-1}-\mu_j+1)\delta_j} M(I_j')$$

$$+ \rho^{\delta_{j+1}-\gamma_{j+1}} M(I_j').$$

Using the fact that for $(k,m) \in S_j$, $k + \delta_j m = \gamma_j$ we can reduce the preceding estimate to

$$Q_j' \in \rho M(I_j') + \rho^{\delta_j} M(I_j') + \rho^{\delta_j-\delta_{j-1}} M(I_j') + \rho^{\delta_j-\gamma_{j+1}} M(I_j').$$

Hence

$$Q_j' \in \rho^{\min(1,\delta_j,\delta_j-\delta_{j-1},\delta_{j+1}-\gamma_{j+1})} M(I_j').$$

Similar reasoning shows that

$$Q_j'' \in \rho^{\min(1,\gamma_j,\gamma_j-\delta_{j-1},\delta_{j+1}-\delta_j)} M(I_j'').$$

We combine these two estimates, observing that

$$\min(1,\delta_j,\delta_j-\delta_{j-1},\delta_{j+1}-\gamma_{j+1},\gamma_j,\gamma_j-\delta_{j-1},\delta_{j+1}-\delta_j)$$

$$= \min(1,\gamma_j-\delta_{j-1},\delta_{j+1}-\gamma_{j+1},\gamma_j) = \Delta_j,$$

and thereby obtain the conclusion of the Theorem in the special case that $0 < j < p$. The two remaining cases follow by slight variants of the above argument which we omit.

Corollary: On I_j, any power of $a(t,\rho)$ has the asymptotic expansion

$$a^k \sim \left[\rho^{\gamma_j}(t\rho^{-\delta_j})^{\mu_j} a_j(t\rho^{-\delta_j})\right]^k \sum_{N=0}^{\infty} \binom{k}{N} Q_j^N .$$

Proof: Since $\Delta_j > 0$, for ρ sufficiently small $|Q_j| < \frac{1}{2}$, which implies the stronger result that the above series is uniformly convergent, in addition to being formally convergent.

The proof of our final result, Theorem 7 below, consists of a constructive procedure for explicitly solving the connection problem. In this construction we require the following notion of a negligible formal series (see van der Corput [4]).

Definition: For fixed j, we say that a formal series $F(\rho,s,s_1,\ldots,s_p,s_1'\ldots s_{j-1},s_{j+1}',\ldots,s_p')$ on J^* is negligible if some positive N

$$\rho^N F \sim \sum_{k=1}^{p} \left\{ F_k(s_k\rho^{\gamma_k-\delta_k}) + F_k^*(s_k^{-1}\rho^{\delta_{k+1}-\gamma_k}) + f_k \log s_k \right\}$$

$$+ \sum_{\substack{k=1 \\ k\neq j}}^{p} \left\{ F_k^{**}(s_k') + f_k^* \log s_k \right\}$$

where $F_k(s)$, $F_k^*(s)$ are formal power series in $s^{1/2}$ without constant form, the F_k^{**} are formal power series in $s^{1/2}$ with only negative exponents, and f_k, f_k^* and the coefficients of F_k, F_k^*, F_k^{**} are proper

formal power series in a fractional power of ρ. The term "negligible" is justified by the following:

Lemma 8A. If F is a function on J^* which depends on (ρ, s_j') alone and

$$F(\rho, s_j') \sim F^{(1)}(\rho, s_j') + F^{(2)}$$

where $F^{(1)}$ is a formally convergent series whose terms are functions of (ρ, s_j') alone and $F^{(2)}$ is negligible, then

$$F^{(2)} \diamond 0.$$

Proof: $F^{(2)}$ can be written in the form

$$F^{(3)} \diamond \sum_{N=1}^{\infty} \rho^{\xi_N} \sum_{k=1}^{p} \Big\{ \Gamma_{Nk}(s_k) + \Gamma_{Nk}^*(s_k') + b_{Nk} \log s_k + b_{Nk}^* \log s_k' \Big\}$$

where Γ_N, Γ_N^* are finite formal power (Laurent) series without constant terms, b_n, b_N^* are constants, $\Gamma_{N_j}^*(s) \equiv b_{N_j}^* = 0$, and the sequence ξ_N is strictly increasing to ∞. It follows easily that $F \sim F^{(1)} + F^{(3)}$ and $F^{(2)} \diamond 0 \Longleftrightarrow F^{(3)} \diamond 0$. The relation $F \sim F^{(1)} + F^{(3)}$ implies

$$\rho^{-\xi_1}(F - F_1^{(1)}) \in \sum_{k=1}^{p} \Big\{ \Gamma_{1k} + \Gamma_{1k}^* + b_{1k} \log s_k + b_{1k}^* \log s_k' \Big\} + \rho^{\xi_2 - \xi_1} M(J^*)$$

where $F_1^{(1)}$ is a suitable partial sum of the formal series $F^{(1)}$.

Since the left hand side depends only on (ρ, s_j'), this inclusion implies

$$\sum_{k=1}^{p} \Big\{ \Gamma_{1k}(u_1, u_p') + \dots + b_{1k}^{*} \log u_k' - \Gamma_{1k}(v_1, v_p') - b_{1k}^{*} \log v_k' \Big\} \in \rho^{\xi_2 - \xi_1} M(J^{*} \times J^{*})$$

which clearly implies $\Gamma_{1k} \equiv \Gamma_{1k}'^{*} \equiv b_{1k} = b_{1k}^{*} = 0$. Finally, induction shows that $\Gamma_{Nk} \equiv \Gamma_{Nk}'^{*} \equiv b_{Nk} = b_{Nk}^{*} = 0$ for all N, establishing the desired result.

<u>Theorem 7</u>. <u>Let</u> $t = s\rho^{\delta_j}$. <u>On</u> I_j <u>the matrix</u> $\underset{\sim}{Y}$ <u>of</u> (7.4) <u>has an asymptotic representation of the form</u>

$$(8.7)\quad \underset{\sim}{Y}(t,p)\exp-\rho^{-n}\begin{pmatrix} c_+^{(j)}(p) & 0 \\ 0 & c_-^{(j)}(p) \end{pmatrix} \exp-\rho^{n}\begin{pmatrix} q_+^{(j)}(s,p) & \\ & q_-^{(j)}(s,p) \end{pmatrix} \sim \underset{\sim}{Q}^{(j)}(s,p)$$

<u>where</u>

1. $\underset{\sim}{Q}^{(j)}$ <u>is a formal matrix power series in</u> $\rho^{\frac{1}{2\sigma_j}}$ <u>with coefficients which are</u> C^{∞} <u>in</u> s <u>for</u> $0 < s < \infty$ <u>if</u> $j > 0$ <u>and for</u> $0 < s \le t_1$ <u>if</u> $j = 0$ <u>and</u> $q_{\pm}^{(j)}(s,\rho)$ <u>is a polynomial in</u> $\rho^{-\frac{1}{2\sigma_j}}$ <u>with similar coefficients</u>.

2. $c_{\pm}^{(j)}$ <u>is a function of</u> ρ <u>alone</u>. <u>Let</u> $\sigma = \mathrm{lcm}(\sigma_1, \dots, \sigma_p)$. $c_{\pm}^{(j)}$ <u>has an asymptotic expansion of the form</u>

(8.8) $$c_{\pm}^{(j)} \sim f_{\pm}^{(j)}(\rho^{+\frac{1}{2\sigma}}) = \log \rho \; g_{\pm}^{(j)}(\rho^{\frac{1}{2\sigma}})$$

where $f_{\pm}^{(j)}$, $g_{\pm}^{(j)}$ are formal power series with coefficients which can be expressed explicitly in terms of integrals of the form (independent of ρ)

$$\int_{0}^{1} a_k^{h/2}(s)s^{i/2}ds \quad , \quad \int_{1}^{\infty} a_k^{h/2}(s)s^{i/2}ds$$

(8.9) $$\int_{0}^{t_1} \left[s^{\mu_0}a_0(s)\right]^{h/2} F(s)ds \quad , \quad F(s) \in C^{\infty}[0,t_1] \; ,$$

or suitably defined finite parts of these integrals in cases where the indicated integral diverges. Here $0 \leq k \leq p$ and h and i are signed integers.

Proof: Let

(8.10) $$c_{\pm}^{(j)} = -\int_{\sigma_j}^{t_1} r_{\pm}(\theta,\rho)d\theta$$

Then (7.3) can be written in the form

(8.11) $$\underset{\sim}{U} \exp-\rho^{-n} \begin{bmatrix} c_{+}^{(j)} & 0 \\ 0 & c_{-}^{(j)} \end{bmatrix} \exp-\rho^{-n} \int_{\sigma_j}^{t} \begin{bmatrix} r_{+} & 0 \\ 0 & r_{-} \end{bmatrix} d\theta = \begin{bmatrix} 1 & 1 \\ r_{+} & r_{-} \end{bmatrix} .$$

1. We show that $r_{\pm}$, $\int_{\sigma_j}^{t} r_{\pm}d\theta$ possess asymptotic series of the kind specified in 1. To show this it is sufficient to show that $R_k(\pm\lambda,t,\rho)$, $\int_{c}^{t} R_k(\pm\lambda,\theta,\rho)d\theta$ possess asymptotic series of the same kind. The recursion formula (5.2) for R_k shows that R_k can

be written as a finite sum of terms of the form $a^{k/2}h(t,\rho)$ where k is a signed integer and $h(t,\rho)$ is C^∞ for $0 \le t \le t_1$, $0 \le \rho \le \rho_1$. Moreover each such h possesses an asymptotic power series in ρ with coefficients in $C^\infty[0,t]$. Hence it is sufficient to show that

$$a^{k/2}(s\rho^{\delta_j},\rho)h(s\rho^{\delta_j}) \ , \int_{\sigma_j}^{t} a^{k/2}(\theta\rho^{\delta_j},\rho)h(\theta)d\theta \ , \quad h \in C^\infty[0,t_1]$$

possess series of the same kind. The integral can be written

$$\rho^{\delta_j}\int_{s_j'}^{s} a^{k/2}(\rho^{\delta_j}\theta,\rho)h(\rho^{\delta_j}\theta)d\theta.$$

By the corollary to Theorem 6

$$a^{k/2}(t,\rho)\sim p^{k/2\gamma_j}\sum_{N=0}^{\infty}\binom{\frac{k}{2}}{N}\left[s^{\mu_j}a_j(s)\right]^{k/2-N}$$

$$\left[a(s\rho^{\delta_j},\rho) - \rho^{\gamma_j}s^{\mu_j}a_j(s)\right]^{N}.$$

But $\left[a(s\rho^{\delta_j},\rho) - \rho^{\gamma_j}s^{\mu_j}a_j(s)\right]^{N} h(\rho^{\delta_j}s)$ is a C^∞ function of s and $p^{\frac{1}{\sigma_j}}$: Since it possesses an asymptotic power series expansion in $\rho^{\frac{1}{\sigma_j}}$ it is sufficient to consider expressions of the form

$$\left[s^{\mu_j}a_j(s)\right]^{k/2} h(s) \ , \int_{s_j'}^{s}\left[s^{\mu_j}a_j(s)\right]^{k/2} h(s)ds \quad h(s) \in \begin{cases} C^\infty(0,t_1] & j=0 \\ C^\infty(0,\infty) & j\neq 0. \end{cases}$$

But s_j' does not depend on ρ. Hence the above expressions are functions of s alone belonging to

$$C^{\infty}(0,\infty) \text{ (or } C^{\infty}(0,t_1] \text{ if } j = 0).$$

These are trivial instances of the kind of asymptotic expansion appearing in our conclusion whereby we conclude that

$r_{\pm}$, $\int_{\sigma_j}^{t} r_{\pm}\,d\theta$ possess asymptotic expansions of the asserted form.

We let $q_{\pm}^{(j)}(s,p)$ be the partial sum of the expansion of $\int_{\sigma_j}^{t} r_{\pm}\,d\theta$ including all terms of order $\leq 2n\,\sigma_j$ in $\rho^{\frac{1}{2\sigma_j}}$. We let $\underset{\sim}{Q}(s,p)$ be the formal series which is the asymptotic expansion of

$$\begin{bmatrix} 1 & 1 \\ r_+ & r_- \end{bmatrix} \exp \rho^{-n} \begin{bmatrix} \int_{\sigma_j}^{t} r_+\,d\theta - q_+^{(j)} & 0 \\ 0 & \int_{\sigma_j}^{t} r_-\,d\theta - q_-^{(j)} \end{bmatrix}.$$

This establishes 1, which asserts that the solutions of $(2.3)_j$ obtained by means of transformation $(2.2)_j$ from the solutions of (1.1) given by

$$\begin{bmatrix} 1 & 1 \\ r_+ & r_- \end{bmatrix} \exp \rho^{-n} \int_{\sigma_j}^{t} \begin{bmatrix} r_+ & 0 \\ 0 & r_- \end{bmatrix} d\theta$$

have asymptotic representations of the form given in $(3.1)_j$. However these expressions are more easily obtained by purely formal procedures.

2. As in the proof of 1, it suffices to show that

$$F(\rho, c_j) = \int_{\sigma_j}^{t_1} a^{N/2}(t,p)h(t)\,dt \quad , \quad h(t) \in C^{\infty}[0,t_1]$$

has an expansion of the asserted form for any signed integer N. We write $F(\rho,\sigma_j)$ in the form

$$(8.12)\qquad F = \rho^{+\delta_j} \int_{s'_j}^{\tau_j \rho^{-\delta_j}} a^{N/2}(\theta\rho^{\delta_j})\, h(\theta\rho^{\delta_j})\, d\theta$$

$$\sum_{k=1}^{j-1} \left\{ \rho^{\delta_k} \int_{\tau_k \rho^{-\delta_k}}^{\sigma_{k-1}} a^{N/2}(\theta\rho^{\delta_k},\rho)\, h(\theta\rho^{\delta_k})\, d\theta \right.$$

$$\left. + \rho^{\delta_k} \int_{\sigma_{k-1}}^{\tau_{k-1}\rho^{-\delta_k}} a^{N/2}(\theta\rho^{\delta_k},\ )\, h(\theta\rho^{\delta_k})\, d\theta \right\}$$

We note the behavior of the limits of integration as $\rho \to 0$, namely $\tau_{k-1}\rho^{-\delta_k} \to \infty$, $\gamma_k \rho^{-\delta_k} \to 0$. The representation (8.12) makes F appear to depend upon the variables $s_{j+1},\dots,s_p$, etc. In fact we could eliminate this formal dependence by making definite numerical choices for s_k, s'_k and r_k. However such a choice would be most unwise since it would force us to compute a great many qualities which have no bearing on the final result. Lemma 8A insures that we may neglect all negligible formal series which appear in asymptotic expansions of the individual terms on the right hand side of (8.12). Thus to establish the conclusion of 2. it suffices to show that each term on the right hand side of (8.12) has an asymptotic expansion which is the sum of a series of the form given in our conclusion and a negligible series.

We again apply the Corollary to Theorem 6 and arguments used in the proof of 1. to conclude that we need only consider integrals of the form

$$\int_{s_k'}^{\tau_k \rho^{-\delta_k}} a_k^{\frac{N_1}{2}}(s) s^{\frac{N_2}{2}} ds\ , \quad \int_{\tau_{k-1}\rho^{-\delta_k}}^{s_k'} a_k^{\frac{N_1}{2}} ds \quad K > 0$$

$$\int_{\tau_1}^{t_1} \left[s^{\mu_o} a_o(s) \right]^{N/2} f(s) ds \quad f(s) \in C^{\infty}[0,t_1].$$

We consider the special case

$$(8.13) \qquad \int_{\tau_{k-1}\rho^{-\delta_k}}^{s_k'} a_k^{\frac{N_1}{2}}(s)\ s^{\frac{N_2}{2}}\ ds$$

where we suppose that the integral

$$(8.14) \qquad \int_0^{s_k'} a_k^{\frac{N_2}{2}}(s)\ s^{\frac{N_2}{2}}\ ds$$

exists. An asymptotic expansion for (8.13) is readily computed by writing the integral in the form

$$\int_{\tau_{k-1}\rho^{-\delta_j}}^{s_k'} = \int_0^{s_k'} - \int_0^{\tau_k \rho^{-\delta_k}} .$$

Inserting the formal power series expansion for the integrand in the second integral shows that its asymptotic expansion is negligible. Hence in this special case it suffices to compute

the integral (8.14). If the integral (8.14) is not convergent we define a finite part of the integral by subtracting a finite power series in $s^{-1/2}$ containing only terms of exponent less than -1. This permits duplication of the same argument except that we must now add to (8.14) the integral from $\tau_k \rho^{-\delta_k}$ to s_j' of the finite power series. Since this integral is of the form C log ρ plus a negligible series we can again draw the conclusion that the integral has an expansion of the desired form which can be expressed in terms of (8.14). Similar arguments apply to integrals of the two remaining forms. Finally making the particular choice $s_k' = 1$ for all k, the last conclusion of the Theorem follows,

Remark: All of our asymptotic statements thus far have been defined by sequences of ideals or the relation $\sim$. We fall back on a vaguer notion of "asymptotic description" in our final statement, which can be easily explicated by making suitable assumptions about the initial values which appear in the following:

<u>Corollary</u> (<u>Asymptotic Solution of the Initial Value Problem</u>). <u>The solution</u> $y(t,\rho)$ <u>and its derivative</u> $y(t,\ )$ <u>specified by the functions (initial values)</u> $y(0,\rho)$, $\dot{y}(0,\rho)$ <u>are described asymptotically on</u> I_j <u>by</u>

$$Q^{(j)}(s,p)\exp\left\{\rho^{-n}\begin{bmatrix} c_+^{(p)}-c_+^{(j)}+q_+^{(j)}(s,p), 0 & \\ 0 \quad , c_-^{(p)}-c_-^{(j)}+q_-^{(j)}(s,p) \end{bmatrix}\right\}$$

$$\cdot\begin{bmatrix} 1 & 1 \\ r_+(0,\rho) & r_-(0,\rho) \end{bmatrix}^{-1}\begin{bmatrix} y(0,\rho) \\ \dot{y}(0,\rho) \end{bmatrix} .$$

Proof: The unique solution specified by initial values is described by the vector (see(7.3))

$$W(t,\rho)W^{-1}(0,\rho)\quad\begin{bmatrix} y(0,\rho) \\ y(0,\rho) \end{bmatrix}$$

which can be written in the form

$$\left\{\begin{bmatrix} 1 & 1 \\ r_+(t,\rho) & r_-(t,\rho) \end{bmatrix} \exp \rho^{-n} \int_{\sigma_j}^{t} \begin{bmatrix} r_+ & 0 \\ 0 & r_- \end{bmatrix} d\theta \right\} \cdot \exp -\rho^{-n} \begin{bmatrix} c_+^{(p)} - c_+^{(j)} \\ c_+^{p} - c_-^{(j)} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 1 \\ r_+(0,\rho) & r_-(0,\rho) \end{bmatrix}^{-1} \begin{bmatrix} y(0,\rho) \\ y^{a}(0,\rho) \end{bmatrix} .$$

Since the asymptotic behavior on I_j of the expression in brackets is described by

$$\underset{\sim}{Q}^{(j)} \exp \rho^{-n} \begin{bmatrix} c_+^{(j)} & \\ & c_-^{(j)} \end{bmatrix}$$

the conclusion follows.

9. What is a Turning Point?

It is easily seen that if any a_j has a real zero ζ, then the condition $t = \rho^{\delta_j} \zeta$ forces our formal procedure to break down, or granting our hypotheses, if $a(t,\rho)$ is holomorphic, this breakdown occurs when ζ is a complex root of a_j. It seems natural to say that this condition describes a turning point phenomenon. A turning point

problem can be regarded, not as an anomaly in the solutions of a differential equation, which are usually well behaved at the turning point, but as a failure of the means which we use to get hold of the asymptotic behavior of solutions. Evidently the possibility of such a failure is conditioned by the means which we have at our disposal. Since the preceding results are obtained by a simple extension of the resources of non-turning point methods (essentially the adjunction of the roots of $\lambda^2 - a(t,\rho)$) and since we have strong evidence that such problems can be treated with satisfactory generality (at least in their formal aspects) we prefer to consider the problem treated above as a non-turning point problem.

However there are some interesting points of contact with turning point theory. We draw a resemblance to comparison methods (see Erdelyi [1]) which exploit the resemblance of a differential equation to a simpler problem for which asymptotic solutions are known. We can well describe the problem treated above as a differential equation having solutions which can be successfully compared to the algebroid function of two variables $\lambda(t,\rho)$. Moreover in the case in which $a(t,\rho)$ is holomorphic, the determination of suitable solutions on complex (t,ρ) domains presents a great wealth of geometric phenomena which do not seem susceptible to general treatment. Indeed investigations of single examples in this respect appear to share the magnitude and flavor of turning point investigations, as a forthcoming study of a specific problem with holomorphic coefficients will show.

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